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Wave propagator via quantum fluid dynamics

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Via the de Broglie–Bohm causal interpretation of quantum mechanics, we develop a protocol to obtain a propagator for the guiding wave function where the features of the quantum potential are kept. Our analysis is extended to include a friction mechanism. [S1063-651X(97)09507-X]

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In the causal interpretation of quantum mechanics, the primary concept is introduced that a particle has a definite path which is determined by a suitable equation of motion and that this path is fundamentally affected by a guiding wave function [1–5]. Accordingly, the connection between the particle and wave properties can be obtained by writing the guiding wave function in the polar form

$$\psi=\phi\,\exp(iS), \tag{1}$$

where

$$\frac{\partial\rho}{\partial t}+\frac{\partial(\rho v)}{\partial x}=0, \tag{2}$$

$$\frac{\partial v}{\partial t}+v\,\frac{\partial v}{\partial x}=-\frac{1}{m}\,\frac{\partial}{\partial x}\,(V+V_{\rm qu}), \tag{3}$$

and

$$\rho=\phi^2, \tag{4}$$

$$v=\frac{\hbar}{m}\,\frac{\partial S}{\partial x}, \tag{5}$$

$$V_{\rm qu}=-\frac{\hbar^2}{2m\phi}\,\frac{\partial^2\phi}{\partial x^2}. \tag{6}$$

Equation (2) represents the conservation of probability with density ρ, whereas Eq. (3) describes paths of a particle with velocity

$$\frac{dx}{dt}=v(x,t)|_{x=x(t)}=\frac{\hbar}{m}\,\frac{\partial S}{\partial x}\bigg|_{x=x(t)}, \tag{7}$$

subject to an arbitrary external potential V and the so-called quantum potential $V_{\rm qu}$ [1–5].

With the help of Eq. (7), we can readily obtain

$$\frac{\partial S}{\partial t}+\left(\frac{\hbar}{2m}\right)\left(\frac{\partial S}{\partial x}\right)^2+\frac{1}{\hbar}\,(V+V_{\rm qu})=0. \tag{8}$$

From Eqs. (7) and (8), we can also obtain

$$\frac{d^2x}{dt^2}=-\frac{1}{m}\,\frac{\partial}{\partial x}\,(V+V_{\rm qu}), \tag{9}$$

where

$$\frac{d}{dt}=\frac{\partial}{\partial t}+v\,\frac{\partial}{\partial x} \tag{10}$$

is the hydrodynamical derivative.

Equation (9) has the form of Newton's second law, in which the particle is subject to a quantum potential $V_{\rm qu}$ in addition to the classical potential V. The classical set of paths is obtained by considering the case when the amplitude of the wave function is a slowly varying function of position, i.e., $V_{\rm qu}\rightarrow 0$.

In what follows, we develop a protocol to obtain a propagator for a wave function by retaining explicitly some of the features of the quantum potential. Therefore, this procedure attempts to generalize that developed by Feynman and Hibbs

[6], since the procedure of Feynman and Hibbs is viewed as a method for obtaining the quantum wave function from the set of classical paths, for which $V_{\rm qu}=0$.

We investigate the quantum hydrodynamical evolution of the wave packet

$$\phi(x,t)=[2\pi a^2(t)]^{-1/4}\exp\left\{-\left[\frac{[x-X(t)]^2}{4a^2(t)}\right]\right\}, \quad (11)$$

where $X(t)$ represents the classical path. To this end, we expand $S(x,t)$, $V(x,t)$, and $V_{\rm qu}(x,t)$ around $X(t)$ up to second order:

$$S(x,t)=S[X(t),t]+S'[X(t),t][x-X(t)]+\frac{S''[X(t),t]}{2}[x-X(t)]^2, \quad (12)$$

$$V(x,t)=V[X(t),t]+V'[X(t),t][x-X(t)]+\frac{V''[X(t),t]}{2}[x-X(t)]^2, \quad (13)$$

$$V_{\rm qu}(x,t)=V_{\rm qu}[X(t),t]+V'_{\rm qu}[X(t),t][x-X(t)]+\frac{V''_{\rm qu}[X(t),t]}{2}[x-X(t)]^2. \quad (14)$$

Next, substituting Eq. (11) into Eq. (2) and integrating, we find

$$v(x,t)=\frac{\dot a}{a}[x-X(t)]+\dot X(t). \quad (15)$$

A connection to Eq. (12) can be established with the help of Eq. (7) by collecting terms in $[x-X(t)]^0$ and $[x-X(t)]$:

$$S'[X(t),t]=\frac{m\dot X}{\hbar}, \quad (16)$$

$$S''[X(t),t]=\frac{m\dot a}{\hbar a}. \quad (17)$$

Now, substituting Eqs. (11)–(17) into Eq. (8) and collecting terms in $[x-X(t)]^0$, $[x-X(t)]$, and $[x-X(t)]^2$, we have

$$\dot S_0=\frac{1}{\hbar}\left(\frac{1}{2}m\dot X^2-V[X(t),t]-\frac{\hbar^2}{4ma^2}\right), \quad (18)$$

$$\ddot X=-\frac{1}{m}V'[X(t),t], \quad (19)$$

$$\ddot a+\left(\frac{1}{m}V''[X(t),t]\right)a=\frac{\hbar^2}{4m^2a^3}, \quad (20)$$

where we have denoted $S_0(t)=S[X(t),t]$. It is worth noticing the presence of the quantum potential $V_{\rm qu}$ in the last terms of Eqs. (18) and (20). These equations have the initial conditions

$$X(0)=x_0, \quad \dot X(0)=v_0,$$

$$a(0)=a_0, \quad \dot a(0)=0,$$

$$S_0(0)=\frac{mv_0x_0}{\hbar}. \quad (21)$$

Now the wave packet described by Eq. (1) can be written as

$$\psi(x,t)=[2\pi a^2(t)]^{-1/4}\exp\left[\left(\frac{im\dot a(t)}{2\hbar a(t)}-\frac{1}{4a^2(t)}\right)\times[x-X(t)]^2\right]\exp\left[\frac{im\dot X(t)}{\hbar}[x-X(t)]+\frac{imv_0x_0}{\hbar}\right]\times\exp\left[\frac{i}{\hbar}\int_0^t dt'\left(\frac{1}{2}m\dot X^2(t')-V[X(t)]-\frac{\hbar^2}{4ma^2(t)}\right)\right]. \quad (22)$$

Next, we turn to finding the propagator $K(x,x_0,t)$ as defined by the integral equation

$$\psi(x,t)=\int_{-\infty}^{+\infty}dx_0K(x,x_0,t)\psi(x_0,0). \quad (23)$$

Let us first define the normalized quantity

$$\Phi(v_0,x,t)=(2\pi a_0^2)^{1/4}\psi(v_0,x,t), \quad (24)$$

which satisfies the completeness relation [7]

$$\int_{-\infty}^{+\infty}dv_0\Phi^*(v_0,x,t)\Phi(v_0,x',t)=(2\pi\hbar/m)\delta(x-x'). \quad (25)$$

From Eq. (2), it follows that

$$\frac{\partial(\Phi^*\psi)}{\partial t}+\frac{\partial(\Phi^*\psi v)}{\partial x}=0, \quad (26)$$

which after integration yields

$$\frac{\partial}{\partial t}\int_{-\infty}^{+\infty}dx\ \Phi^*\psi=0, \quad (27)$$

whence

$$\int_{-\infty}^{+\infty}dx'\ \Phi^*(v_0,x',t)\psi(x',t)=\int_{-\infty}^{+\infty}dx_0\Phi^*(v_0,x_0,0)\psi(x_0,0). \quad (28)$$

Multiplying Eq. (28) by $\Phi(v_0,x,t)$, integrating with respect to v_0, and using Eq. (25), we have

$$\psi(x,t)=(m/2\pi\hbar)\int_{-\infty}^{+\infty}dv_0 \times\Phi(v_0,x,t)\int_{-\infty}^{+\infty}dx_0\Phi^*(v_0,x_0,0)\psi(x_0,0), \tag{29}$$

whence the propagator reads

$$K(x,x_0,t)=(m/2\pi\hbar)\int_{-\infty}^{+\infty}dv_0\Phi(v_0,x,t)\Phi^*(v_0,x_0,0). \tag{30}$$

With the help of Eqs. (21), (22), and (24), we have explicitly

$$K(x,x_0,t)=(m/2\pi\hbar)\int_{-\infty}^{+\infty}dv_0\left(\frac{a(t)}{a_0}\right)^{-1/2} \times\exp\left[\left(\frac{im\dot{a}(t)}{2\hbar a(t)}-\frac{1}{4a^2(t)}\right) \times[x-X(t)]^2+\frac{im\dot{X}(t)}{\hbar} \times[x-X(t)]\right]\exp\left[\frac{i}{\hbar}\int_0^t dt'\left(\frac{1}{2}m\dot{X}^2(t') -V[X(t')]-\frac{\hbar^2}{4ma^2(t')}\right)\right], \tag{31}$$

where $X(t)$ and $a(t)$ are the solutions to Eqs. (19) and (20), subject to the initial conditions (21). From Eqs. (12) and (31), the zeroth-order quantum action (divided by $\hbar$)

$$S_0[X(t),t]=\frac{1}{\hbar}\int_0^t dt'\left(\frac{1}{2}m\dot{X}^2(t')-V[X(t')] -V_{\rm qu}[X(t')]\right) \tag{32}$$

shows the presence of the quantum potential [8]

$$V_{\rm qu}[X(t)]=\frac{\hbar^2}{4ma^2(t)}. \tag{33}$$

Moreover, Eqs. (30) and (31) show that the relevant quantum-mechanical information for the propagator is contained in the guiding wave function. The quantum propagator can be viewed as an expansion of the guiding wave function over the v_0 space [9].

Next, we turn to the description of a (Ohmic) friction mechanism via the causal interpretation of quantum mechanics. This description is motivated by classical phenomenological models that have successfully resolved problems of, for example, a particle in a gas, without having to account for the Avogadro's number of degrees of freedom of the environment, although in a different context we have the example of the Navier-Stokes model for the description of a viscous fluid [10].

To this end, we generalize Eq. (3) to [11]

$$\frac{\partial v}{\partial t}+v\,\frac{\partial v}{\partial x}+\nu v=-\frac{1}{m}\,\frac{\partial}{\partial x}\,(V+V_{\rm qu}), \tag{34}$$

where ν is the friction coefficient and the third term on the left-hand side of Eq. (34) accounts for the Ohmic friction mechanism. With the help of Eq. (7), we can readily obtain

$$\frac{\partial S}{\partial t}+\nu S+\left(\frac{\hbar}{2m}\right)\left(\frac{\partial S}{\partial x}\right)^2+\frac{1}{\hbar}\,(V+V_{\rm qu})=0. \tag{35}$$

Equations (7) and (35) yield

$$\frac{d^2x}{dt^2}+\nu\,\frac{dx}{dt}=-\frac{1}{m}\,\frac{\partial}{\partial x}\,(V+V_{\rm qu}). \tag{36}$$

By following the same procedure developed to obtain Eqs. (18)–(20), we arrive at

$$\dot{S}_0+\nu S_0=\frac{1}{\hbar}\left(\frac{1}{2}\,m\dot{X}^2-V[X(t),t]-\frac{\hbar^2}{4ma^2}\right), \tag{37}$$

$$\ddot{X}+\nu\dot{X}=-\frac{1}{m}\,V'[X(t),t], \tag{38}$$

$$\ddot{a}+\nu\dot{a}+\left(\frac{1}{m}\,V''[X(t),t]\right)a=\frac{\hbar^2}{4m^2a^3}, \tag{39}$$

subject to the same initial conditions as in Eq. (21).

Likewise, the wave packet described by Eq. (1) can be written as

$$\psi(x,t)=[2\pi a^2(t)]^{-1/4}\exp\left[\left(\frac{im\dot{a}(t)}{2\hbar a(t)}-\frac{1}{4a^2(t)}\right)[x-X(t)]^2\right] \times\exp\left[\frac{im\dot{X}(t)}{\hbar}\,[x-X(t)]+\frac{imv_0x_0}{\hbar}\right] \times\exp\left[\frac{i}{\hbar}\,e^{-\nu t}\int_0^t dt' \times e^{\nu t'}\left(\frac{1}{2}\,m\dot{X}^2(t')-V[X(t)]-\frac{\hbar^2}{4ma^2(t)}\right)\right]. \tag{40}$$

Because our dissipative model keeps the conservation of probability Eq. (2), Eqs. (24)–(30) hold true. Consequently, we write the dissipative propagator as

$$K(x,x_0,t)=(m/2\pi\hbar)\int_{-\infty}^{+\infty}dv_0\left(\frac{a(t)}{a_0}\right)^{-1/2} \times\exp\left[\left(\frac{im\dot{a}(t)}{2\hbar a(t)}-\frac{1}{4a^2(t)}\right)[x-X(t)]^2 +\frac{im\dot{X}(t)}{\hbar}\,[x-X(t)]\right]\exp\left[\frac{i}{\hbar}\,e^{-\nu t}\int_0^t dt' \times e^{\nu t'}\left(\frac{1}{2}\,m\dot{X}^2(t')-V[X(t')]-\frac{\hbar^2}{4ma^2(t')}\right)\right], \tag{41}$$

where $X(t)$ and $a(t)$ are the solutions to Eqs. (38) and (39), subject to the initial conditions (22).

The classical part of the zeroth-order quantum action (divided by $\hbar$) in the propagator above,

$$S_0^{\mathrm{cl}}[X(t),t]=\frac{1}{\hbar}\,e^{-\nu t}\int_0^t dt'\;e^{\nu t'}\left(\frac{1}{2}\,m\dot{X}^2(t')-V[X(t')]\right), \tag{42}$$

can be contrasted to that introduced by Caldirola and Kanai [12,13],

$$S_{\mathrm{CK}}=\frac{1}{\hbar}\int_0^t dt'\;e^{\nu t'}\left(\frac{1}{2}\,m\dot{X}^2(t')-V[X(t')]\right). \tag{43}$$

As pointed out elsewhere [14], this action does not describe dissipation correctly, but rather a system with a time-varying mass, since the dissipative exponential factor can be transformed away [15].

Finally, from Eqs. (2) and (34), we can construct the relation [11].

$$\frac{\partial U}{\partial t}+\frac{\partial Q}{\partial x}=-\nu\rho v^2, \tag{44}$$

where

$$U=\rho[\tfrac{1}{2}mv^2+V+V_{\mathrm{qu}}] \tag{45}$$

and

$$Q=vU+\frac{\hbar^2}{2m^2}\left[\sqrt{\rho}\,\frac{\partial^2\sqrt{\rho}}{\partial x\partial t}-\frac{\partial\sqrt{\rho}}{\partial t}\,\frac{\partial\sqrt{\rho}}{\partial x}\right] \tag{46}$$

are the energy density and energy density flux, respectively.

The preceding equations demonstrate that the energy-dissipation theorem is correctly satisfied and validate the model developed in this work towards a further understanding of the role of dissipation in quantum mechanics. Temperature can enter in the formalism through a stochastic force added to the right side of Eq. (34). Via the fluctuation-dissipation theorem, a solution to Eq. (38) can be obtained and incorporated to the wave propagator (41).

Above all, this work shows that the relevant quantum-mechanical information for the propagator is contained in the guiding wave function. The quantum propagator can be viewed as an expansion of the guiding wave function over the v_0 space. Therefore, it poses an alternative route that suggests further investigations. Applications of this work are in progress and will be published in a forthcoming paper.

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[8] The quantum potential assumes a more general form
$$V_{\mathrm{qu}}(x)=-\frac{\hbar^2}{2m}\left[\frac{[x-X(t)]^2}{4a^4(t)}-\frac{1}{2a^2(t)}\right].$$

[9] The free-particle propagator can be obtained by using $a(t)=a_0\sqrt{1+(t/\tau)^2}$, where $\tau=2ma_0^2/\hbar$ and $X(t)=x_0+v_0t$. Useful formulas are $\exp[-i\tan^{-1}(t/\tau)]=[1-i(t/\tau)]/\sqrt{1+(t/\tau)^2}$ and $\int_0^\infty dy\,\exp[-Ay^2+By]=\sqrt{(\pi/A)}\,\exp(B^2/4A]$.

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